

Criteria of convergence of a non ordinary random continued fractions on a symmetric cone

(Running title: **random continued fractions**)

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Abstract In this paper, we use a notion of ratio based on a division algorithm, to extend to a symmetric cone the definition of a continued fraction in its more general form. We then give a criteria of convergence of a non ordinary random continued fraction that has arisen in the study of some probability distributions related to the beta distribution on the cone of positive definite symmetric matrices or on any symmetric cone.

Keywords: Symmetric matrices, Jordan algebra, symmetric cone, division algorithm, continued fractions.

AMS Classification : 40A15, 17C65, 60B20

1 Introduction

Besides the well known theory of real continued fractions and its applications, many multivariate versions have been introduced to bring answers to some needs which have arisen in different areas. The most important seem to be the ones defined on some matrix spaces. Since there is no single way to define a matrix inverse, there exist many ways to define continued fractions with matrix argument.

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A first natural way is to assume invertibility and use the classical matrix inverse to define a ratio of matrices. A second way is based on the partial matrix inverse. Another way uses the generalized inverse considered to be more efficient in matrix continued fraction interpolation problems. For details about these different matrix continued fractions and their applications, we refer the reader to [5] and to the references within. It is worth pointing out that for the applications, the convergence of a matrix continued fraction is usually needed. However even if in many cases, a three term recurrence relation is available, because of the non-commutativity, it is difficult to derive applicable convergence criteria. The present work is motivated by some probability problems concerning the characterization of distributions such as the generalized beta distribution or the beta-hypergeometric distribution on the cone of positive symmetric matrices or more generally on any symmetric cone. Of course in the space of symmetric matrices, we can no more use the multiplication by the inverse to define a ratio and we need to use a division algorithm. The results will be given in the general setting of a Jordan algebra and its symmetric cone, however to make the paper accessible for a reader who is not familiar with the theory of Jordan algebra, we have emphasized on the case of symmetric matrices. We mention here that, for the proof of a characterization result concerning the Wishart distribution on a symmetric cone, Evelyn Bernadac [1] has used an ordinary continued fraction defined with a construction process based on the inversion without any use of multiplication or ratio. Assuming the independence of the random variables and some conditions on their distributions, she has given a criteria of convergence for the ordinary continued fraction. As a continuation of this work, we first use a notion of quotient based on a division algorithm to define in its more general form, a continued fraction $K(x_n/y_n)$ where (x_n) and (y_n) are two sequences of elements of the symmetric cone. An ordinary continued fraction corresponds to the case where $x_n = e$, for all n , where e is the identity element of the algebra. We then show that, as in the classical matrix case (see [4]), any non ordinary continued fraction is equivalent to an ordinary one. This result has a mathematical interest, but it has no implication on the convergence of the non ordinary continued fraction. In fact, when the sequences in the non ordinary fraction are constituted of independent random variables, the corresponding ordinary continued fraction has not this property. Accordingly, the main part of the paper is devoted to establish a criteria of convergence for the non ordinary random continued fractions of the form $K(x_n/e)$, where (x_n) is a sequence of independent random variables valued in the symmetric cone. This kind of non ordinary random continued fractions with the ordinary ones cover all the cases that arise in the characterization of the most common distributions on symmetric matrices or more generally on any symmetric cone.

2 Symmetric cones

It is well known that any symmetric cone is associated to a Jordan algebra, so that in order to present our results in their most general form, it is necessary to review some facts concerning Jordan algebra and their symmetric cones. Our notations are the ones used in [2] or in [3]. Let us recall that a Euclidean Jordan algebra is a Euclidean space V with scalar product $\langle x, y \rangle$ and a bilinear map

$$V \times V \longrightarrow V, (x, y) \longmapsto x.y$$

called Jordan product such that, for all x, y, z in V ,

- i) $x.y = y.x$,
- ii) $\langle x, y.z \rangle = \langle x.y, z \rangle$,
- iii) there exists e in V such that $e.x = x$,
- iv) $x.(x^2.y) = x^2.(x.y)$, where we used the abbreviation $x^2 = x.x$.

A Euclidean Jordan algebra is said to be simple if it does not contain a nontrivial ideal.

For the algebra of symmetric matrices, the Jordan product is defined by $x.y = \frac{1}{2}(xy + yx)$ where xy is the ordinary product of the matrices x and y and the scalar product is defined by $\langle x, y \rangle = \text{trace}(xy)$.

Now, to each Euclidean simple Jordan algebra V , we attach the set of Jordan squares

$$\overline{\Omega} = \{x^2; x \in V\}.$$

Its interior is denoted Ω .

In general, Ω is a symmetric cone, i.e., a convex cone which is

- i) self dual, i.e., $\Omega = \{x \in V; \langle x, y \rangle > 0 \forall y \in \overline{\Omega} \setminus \{0\}\}$.
- ii) homogeneous, i.e., the group $G(\Omega)$ of linear automorphisms of Ω acts transitively on Ω .

Observe that when V is the algebra of symmetric matrices, Ω (resp. $\overline{\Omega}$) denotes the cone of symmetric positive definite (resp. nonnegative) matrices.

By analogy with the case where $V = \mathbb{R}$ and $\Omega =]0, \infty[$, for x and y in the cone Ω , we write $0 < x < y$ if $y - x \in \Omega$. With this, one can talk about an increasing or decreasing sequence in Ω .

Given a Euclidean simple Jordan algebra V , we denote by G the connected component of the identity in $G(\Omega)$. If V is the space of (r, r) -symmetric matrices and $GL(\mathbb{R}^r)$ is the group of invertible matrices, The elements of $G(\Omega)$ are the maps $g : V \longrightarrow V$ such that there exists a in $GL(\mathbb{R}^r)$ with

$$g(x) = axa^*,$$

where a^* is the transpose of a .

For two automorphisms g_1 and g_2 in G , we write $g_1 g_2$ the composition of g_1 and

g_2 , and for g_1, \dots, g_n in G , we write $\prod_{i=1}^n g_i = g_1 \dots g_n$.

Throughout, we will also use the following notations for $(x, y, z) \in V$:

$$L(x)y = xy$$

$$P(x)y = 2x(xy) - x^2y \text{ (quadratic representation)}$$

$$x \square y = L(xy) + [L(x), L(y)] = L(xy) + L(x)L(y) - L(y)L(x).$$

An element c of V is said to be idempotent if $c^2 = c$. It is a primitive idempotent if furthermore $c \neq 0$ and is not the sum $t + u$ of two non-null idempotents t and u such that $t.u = 0$.

A Jordan frame is a set $\{c_1, \dots, c_r\}$ of primitive idempotents such that $\sum_{i=1}^r c_i = e$ and $c_i.c_j = \delta_{ij}c_i$, for $1 \leq i, j \leq r$. The size r of such a frame is a constant called the rank of V . For any element x of a Euclidean simple Jordan algebra, there exist a Jordan frame $(c_i)_{1 \leq i \leq r}$ and $(\lambda_1, \dots, \lambda_r)$ in \mathbb{R}^r such that $x = \sum_{i=1}^r \lambda_i c_i$. The real numbers $\lambda_1, \lambda_2, \dots, \lambda_r$ depend only on x . They are called the eigenvalues of x and this decomposition is called its spectral decomposition. The trace and the determinant of x are then respectively defined by $\text{tr}(x) = \sum_{i=1}^r \lambda_i$ and $\Delta(x) = \prod_{i=1}^r \lambda_i$.

If c is a primitive idempotent of V , the only possible eigenvalues of $L(c)$ are $0, \frac{1}{2}$ and 1 . The corresponding eigenspaces are respectively denoted by $V(c, 0)$, $E(V, \frac{1}{2})$ and $V(c, 1)$ and the decomposition

$$V = V(c, 0) \oplus V(c, \frac{1}{2}) \oplus V(c, 1)$$

is called the Peirce decomposition of V with respect to c .

Suppose now that $(c_i)_{1 \leq i \leq r}$ is a Jordan frame in V and for $1 \leq i, j \leq r$, let

$$V_{ij} = \begin{cases} V(c_i, 1) = \mathbb{R}c_i & \text{if } i = j \\ V(c_i, \frac{1}{2}) \cap V(c_j, \frac{1}{2}) & \text{if } i \neq j. \end{cases}$$

Then (See [2] Th.IV.2.1) we have $V = \bigoplus_{i \leq j} V_{ij}$ and the dimension of V_{ij} is, for $i \neq j$, a constant d called the Jordan constant. It is related to the dimension n and the rank r of V by the relation $n = r + r(r-1)\frac{d}{2}$. When V is the algebra of symmetric matrices, $d = 1$.

If c is an idempotent and if z is an element of $V(c, \frac{1}{2})$,

$$\tau_c(z) = \exp(2z \square c)$$

is called a Frobenius transformation. It is an element of the group G .

Given a Jordan frame $(c_i)_{1 \leq i \leq r}$, the subgroup of G

$$T = \left\{ \tau_{c_1}(z^{(1)}) \cdots \tau_{c_{r-1}}(z^{(r-1)}) P \left(\sum_{i=1}^r a_i c_i \right), a_i > 0, z^{(j)} \in \bigoplus_{k=j+1}^r V_{jk} \right\}$$

is called the triangular group corresponding to the Jordan frame $(c_i)_{1 \leq i \leq r}$.

It is an important result (see [2], p.113) that for an element y in the symmetric cone Ω there exists a unique element in the triangular group T such that $y = t(e)$. To define a notion of "quotient" in the symmetric cone Ω , we usually introduce a division algorithm which is a measurable map g from Ω into G such that, for all $y \in \Omega$, $g(y)(y) = e$. With the algorithm g , the "quotient" of an element x by y is defined as $g(y)(x)$. The most usual division algorithms are the one corresponding to the quadratic representation and the one corresponding to the Cholesky decomposition. In fact, one can define the quotient of x by y as $P(y^{-\frac{1}{2}})x$. On the other hand, given that for each y in Ω , there exists a unique t in the triangular group T such that $y = t(e)$, we define the multiplication of x by y as

$$\pi(y)(x) = t(x),$$

and the map $y \mapsto t^{-1}$ from Ω into G realizes a division algorithm so that the "quotient" of x by y is defined as

$$\pi^{-1}(y)(x) = t^{-1}(x).$$

We also set

$$\pi^*(y)(x) = t^*(x) \text{ and } \pi^{*-1}(y)(x) = t^{*-1}(x).$$

The definition of $\pi(y)$ and $\pi^*(y)$ may be extended to $y \in -\Omega$, by setting

$$\pi(-y) = -\pi(y) \text{ and } \pi^*(-y) = -\pi^*(y).$$

Consider the set

$$Str(V) = \{g \in GL(V) ; P(g(x)) = gP(x)g^*\}.$$

It is shown in [2] page 150 that $G(\Omega)$ is a subgroup of $Str(V)$. Hence we have in particular that for t in the triangular group T ,

$$P(t(e)) = tP(e)t^* = tt^*. \tag{1}$$

If $y = t(e)$, then

$$P(y) = \pi(y)\pi^*(y). \tag{2}$$

When Ω is the cone of positive definite symmetric matrices, the Cholesky decomposition of an element y of Ω , consists in writing y in a unique manner as $y = tt^*$, where t is a lower triangular matrix with strictly positive diagonal, and we have

$$\pi(y)(x) = txt^* \quad \pi^{-1}(y)(x) = t^{-1}xt^{*-1}, \text{ and } P(y)(x) = yxy,$$

where the last product is the ordinary product of matrices.
In this case, (2) is easily established.

3 Continued fraction in a symmetric cone

We use the division algorithm defined by the Cholesky decomposition. Let $(x_n)_{n \geq 1}$ and $(y_n)_{n \geq 0}$ be two sequences in Ω . The continued fraction denoted

$$K(x_n/y_n)$$

or

$$y_0 + \left[\frac{x_1}{y_1}, \frac{x_2}{y_2}, \dots \right]$$

is an expression whose the n th convergent

$$R_n = y_0 + \left[\frac{x_1}{y_1}, \dots, \frac{x_n}{y_n} \right]$$

is defined in the following recursive way.

$$\begin{aligned} \left[\frac{x_1}{y_1} \right] &= (\pi^{*-1}(x_1)y_1)^{-1} = \pi(x_1)(y_1^{-1}) \\ \left[\frac{x_1}{y_1}, \dots, \frac{x_{n+1}}{y_{n+1}} \right] &= \left(\pi^{*-1}(x_1) \left(y_1 + \left[\frac{x_2}{y_2}, \dots, \frac{x_{n+1}}{y_{n+1}} \right] \right) \right)^{-1} \\ &= \pi(x_1) \left(y_1 + \left[\frac{x_2}{y_2}, \dots, \frac{x_{n+1}}{y_{n+1}} \right] \right)^{-1}. \end{aligned}$$

Note that to get R_{n+1} from R_n , we replace y_n by $y_n + (\pi^{*-1}(x_{n+1})y_{n+1})^{-1}$.

When $x_n = e$, for all n , we say that the continued fraction $K(e/y_n)$ is an ordinary continued fraction. Its k th convergent is given by

$$R_k = y_0 + \left(y_1 + \left(y_2 + \left(y_3 + \dots + (y_k)^{-1} \right)^{-1} \dots \right)^{-1} \right)^{-1}. \quad (3)$$

As mentioned in the introduction, the construction process of this continued fraction uses only the inversion without any use of the product or of the division algorithm.

Next, we show that any continued fraction $K(x_n/y_n)$ is equivalent to an ordinary continued fraction $K(e/a_n)$.

Proposition 3.1 *The continued fraction denoted $K(x_n/y_n)$ is equivalent to the continued fraction $K(e/a_n)$ where*

$$\begin{aligned} a_0 &= y_0 \\ a_{2k} &= \pi(x_1)\pi^{*-1}(x_2)\dots\pi(x_{2k-1})\pi^{*-1}(x_{2k})y_{2k} \end{aligned} \quad (4)$$

$$a_{2k+1} = \pi^{*-1}(x_1)\pi(x_2)\dots\pi(x_{2k})\pi^{*-1}(x_{2k+1})y_{2k+1}. \quad (5)$$

Proof. The proof is performed by induction. We will use the fact that

$$\pi^{-1}(u)v^{-1} = (\pi^*(u)v)^{-1} \quad \text{or equivalently} \quad \pi^*(u)v^{-1} = (\pi^{-1}(u)v)^{-1} \quad (6)$$

Since

$$\left[\frac{x_1}{y_1} \right] = (\pi^{*-1}(x_1)y_1)^{-1},$$

and

$$\begin{aligned} \left[\frac{x_1}{y_1}, \frac{x_2}{y_2} \right] &= \left(\pi^{*-1}(x_1) \left(y_1 + \left[\frac{x_2}{y_2} \right] \right) \right)^{-1} \\ &= \left(\pi^{*-1}(x_1) \left(y_1 + (\pi^{*-1}(x_2)y_2)^{-1} \right) \right)^{-1} \\ &= \left(\pi^{*-1}(x_1)y_1 + \pi^{*-1}(x_1)(\pi^{*-1}(x_2)y_2)^{-1} \right)^{-1} \\ &= \left(\pi^{*-1}(x_1)y_1 + (\pi(x_1)\pi^{*-1}(x_2)y_2)^{-1} \right)^{-1}. \end{aligned}$$

We have that

$$\begin{aligned} a_0 &= y_0 \\ a_1 &= \pi^{*-1}(x_1)y_1 \\ a_2 &= \pi(x_1)\pi^{*-1}(x_2)y_2. \end{aligned}$$

Therefore (5) is true for $k = 0$ and (4) is true for $k = 1$.

Suppose that

$$R_{2k} = a_0 + \left(a_1 + \left(a_2 + \left(a_3 + \dots + (a_{2k})^{-1} \right)^{-1} \dots \right)^{-1} \right)^{-1}.$$

with the a_i , $0 \leq i \leq 2k$, defined by (5) and (4). In particular

$$a_{2k} = \pi(x_1)\pi^{*-1}(x_2)\dots\pi(x_{2k-1})\pi^{*-1}(x_{2k})y_{2k}.$$

To get R_{2k+1} from R_{2k} , we replace in a_{2k} , y_{2k} by $y_{2k} + (\pi^{*-1}(x_{2k+1})y_{2k+1})^{-1}$, that is we replace a_{2k} by

$$\left(a_{2k} + \pi(x_1)\pi^{*-1}(x_2)\dots\pi(x_{2k-1})\pi^{*-1}(x_{2k}) (\pi^{*-1}(x_{2k+1})y_{2k+1})^{-1}\right)^{-1}.$$

Using (6), this can be written

$$\left(a_{2k} + (\pi^{*-1}(x_1)\pi(x_2)\dots\pi^{*-1}(x_{2k-1})\pi(x_{2k})\pi^{*-1}(x_{2k+1})y_{2k+1})^{-1}\right)^{-1}.$$

Hence

$$R_{2k+1} = a_0 + \left(a_1 + \left(a_2 + \left(a_3 + \dots + \left(a_{2k} + (a_{2k+1})^{-1}\right)^{-1}\right)^{-1} \dots\right)^{-1}\right)^{-1}.$$

with

$$a_{2k+1} = \pi^{*-1}(x_1)\pi(x_2)\dots\pi^{*-1}(x_{2k-1})\pi(x_{2k})\pi^{*-1}(x_{2k+1})y_{2k+1}.$$

A similar reasoning is used for R_{2k+2} . ■

4 Convergence of a continued fraction

In this section, we suppose that $(x_n)_{n \geq 1}$ and $(y_n)_{n \geq 0}$ are sequences of independent random variables defined on the same probability space and valued in the cone Ω . As we have mentioned, a criteria of convergence for the ordinary random continued fraction $K(e/y_n)$ has been given in [1]. In what follows, we give a criteria of convergence for the non ordinary random continued fraction $K(x_n/e)$. For the sake of simplicity, $[\frac{x_1}{e}, \frac{x_2}{e}, \dots, \frac{x_k}{e}]$ will be written $[x_1, x_2, \dots, x_k]$ which is the notation usually used for the ordinary continued fraction given in (4). This should not cause any confusion, the continued fractions are completely different, the results and their proofs are different although there are some similarities.

Proposition 4.1 *Let $(x_n)_{n \geq 1}$ be a sequence in Ω . Then for $k \geq 1$,*

$$(-1)^{k+1} ([x_1, x_2, \dots, x_k] - [x_1, x_2, \dots, x_{k+1}])$$

is in Ω .

Proof. The proof uses induction on k .

The property is true for $k = 1$, in fact

$$\begin{aligned} [x_1] - [x_1, x_2] &= x_1 - \pi(x_1)(e + x_2)^{-1} \\ &= \pi(x_1) \left(e - (e + x_2)^{-1} \right) \\ &= \pi(x_1) \left((e + x_2 - e)(e + x_2)^{-1} \right), \end{aligned}$$

which is in $(-1)^2\Omega$. Suppose that the property is true for k . Then

$$\begin{aligned} [x_1, \dots, x_{k+1}] - [x_1, \dots, x_{k+2}] &= \pi(x_1)(e + [x_2, \dots, x_{k+1}])^{-1} - \pi(x_1)(e + [x_2, \dots, x_{k+2}])^{-1} \\ &= \pi(x_1) \left((e + [x_2, \dots, x_{k+1}])^{-1} - (e + [x_2, \dots, x_{k+2}])^{-1} \right) \end{aligned}$$

Using the induction hypothesis, we have that

$$[x_2, \dots, x_{k+1}] - [x_2, \dots, x_{k+2}] \in (-1)^{k+1}\Omega,$$

so that

$$(e + [x_2, \dots, x_{k+1}]) - (e + [x_2, \dots, x_{k+2}]) \in (-1)^{k+1}\Omega.$$

To finish the proof, we use the fact that for all x and y in Ω ,

$$(x - y \in \Omega) \Leftrightarrow (x^{-1} - y^{-1} \in -\Omega).$$

■

Now, Denoting

$$w_k = (-1)^{k+1} ([x_1, x_2, \dots, x_k] - [x_1, x_2, \dots, x_{k+1}]), \quad (7)$$

it is easy to see that

$$[x_1, x_2, \dots, x_n] = x_1 + \sum_{k=1}^{n-1} (-1)^k w_k. \quad (8)$$

Thus the continued fraction $[x_1, x_2, \dots, x_n]$ converges if and only if the alternating series $\sum_{k=1}^{+\infty} (-1)^k w_k$ converges. For the convergence of an alternating series in the algebra V , we have the following result which appears in [1].

Proposition 4.2 *Let $(u_n)_{n \geq 1}$ be a sequence in Ω . If (u_n) is decreasing and converges to 0, then the alternating series $\sum_{k=1}^{+\infty} (-1)^k u_k$ is convergent*

Next, we state and prove a theorem which is crucial for the study of the convergence of the alternating series $\sum_{k=1}^{+\infty} (-1)^k w_k$.

Theorem 4.3 *Let $(x_n)_{n \geq 1}$ be a sequence in Ω , and denote*

$$F_k(x_1, \dots, x_{k+2}) = \left([x_1, \dots, x_k]^{-1} - [x_1, \dots, x_{k+1}]^{-1} \right)^{-1} + \left([x_1, \dots, x_{k+1}]^{-1} - [x_1, \dots, x_{k+2}]^{-1} \right)^{-1}. \quad (9)$$

Then we have

$$F_1 = \pi(x_1)\pi^{*-1}(x_2)\pi^{*-1}(x_3)(e), \quad (10)$$

$$F_2(x_1, x_2, x_3, x_4) = -\pi(x_1)\pi^{*-1}(x_2)\pi^{*-1}(x_3)P(e + \pi^*(x_3)(e))\pi^{*-1}(x_4)(e), \quad (11)$$

and for all $k \geq 3$,

$$F_k(x_1, \dots, x_{k+2}) = (-1)^{k+1}\pi(x_1) \left(\prod_{i=2}^{k-1} \pi^{*-1}(x_i)P(e + [x_{i+1}, \dots, x_k]) \right) \pi^{*-1}(x_k)\pi^{*-1}(x_{k+1})P(u_k)\pi^{*-1}(x_{k+2})(e), \quad (12)$$

where

$$\begin{aligned} u_k &= u_k(x_1, \dots, x_{k+1}) \\ &= e + \pi^*(x_{k+1}) \left(e + \sum_{i=0}^{k-3} (-1)^{i+1} \left(\prod_{j=0}^i \left(\pi^*(x_{k-j})P(e + [x_{k-j}, \dots, x_k])^{-1} \right) \right) (e + [x_{k-i}, \dots, x_k]) \right) \end{aligned}$$

Proof.

$$\begin{aligned} F_1 &= F_1(x_1, x_2, x_3) \\ &= \left([x_1]^{-1} - [x_1, x_2]^{-1} \right)^{-1} + \left([x_1, x_2]^{-1} - [x_1, x_2, x_3]^{-1} \right)^{-1} \\ &= \left(\pi^{*-1}(x_1)(e) - \pi^{*-1}(x_1)(e + x_2) \right)^{-1} \\ &+ \left(\pi^{*-1}(x_1)(e + x_2) - \pi^{*-1}(x_1) \left(e + (\pi^{*-1}(x_2)(e + x_3))^{-1} \right) \right)^{-1} \\ &= \left(-\pi^{*-1}(x_1)(x_2) \right)^{-1} + \left(\pi^{*-1}(x_1)(x_2) - \pi^{*-1}(x_1)\pi(x_2)((e + x_3)^{-1}) \right)^{-1} \\ &= -\pi(x_1)\pi^{*-1}(x_2)(e) + \pi(x_1)\pi^{*-1}(x_2) \left((e - (e + x_3)^{-1})^{-1} \right) \\ &= \pi(x_1)\pi^{*-1}(x_2) \left((e - (e + x_3)^{-1})^{-1} - e \right) \\ &= \pi(x_1)\pi^{*-1}(x_2)\pi^{*-1}(x_3)(e) \end{aligned}$$

Now to calculate

$$\begin{aligned} F_2 &= F_2(x_1, x_2, x_3, x_4) \\ &= \left([x_1, x_2]^{-1} - [x_1, x_2, x_3]^{-1} \right)^{-1} + \left([x_1, x_2, x_3]^{-1} - [x_1, x_2, x_3, x_4]^{-1} \right)^{-1}, \end{aligned}$$

we write

$$[x_1, x_2] = [X_1]$$

with

$$X_1 = \pi(x_1)(e + x_2)^{-1},$$

and we write

$$[x_1, x_2, x_3] = [X_1, X_2]$$

with

$$X_2 = -\pi^* \left((e + x_2)^{-1} \right) \pi(x_2) \left(x_3 (e + x_3)^{-1} \right),$$

finally, we write

$$[x_1, x_2, x_3, x_4] = [X_1, X_2, X_3],$$

with

$$X_3 = \pi^* \left((e + \pi^*(x_3))^{-1} \right) (x_4).$$

Thus using (10), we have

$$\begin{aligned} F_2(x_1, x_2, x_3, x_4) &= F_1(X_1, X_2, X_3) \\ &= \pi(X_1)\pi^{*-1}(X_2)\pi^{*-1}(X_3)(e). \end{aligned}$$

Replacing, we get

$$\begin{aligned} F_2(x_1, x_2, x_3, x_4) &= -\pi(x_1)\pi \left((e + x_2)^{-1} \right) \pi^{-1} \left((e + x_2)^{-1} \right) \pi^{*-1}(x_2) \\ &\quad \pi^{*-1} \left(x_3 (e + x_3)^{-1} \right) \pi^{-1} \left((e + \pi^*(x_3))^{-1} \right) \pi^{*-1}(x_4)(e). \end{aligned}$$

Given that

$$\pi^{*-1} \left(x_3 (e + x_3)^{-1} \right) = \pi^{*-1}(x_3) \pi^{*-1} \left((e + \pi^*(x_3)(e))^{-1} \right),$$

and that

$$\pi^{*-1} \left((e + \pi^*(x_3)(e))^{-1} \right) \pi^{-1} \left((e + \pi^*(x_3)(e))^{-1} \right) = P(e + \pi^*(x_3)(e)),$$

we obtain

$$F_2(x_1, x_2, x_3, x_4) = -\pi(x_1)\pi^{*-1}(x_2)\pi^{*-1}(x_3)P(e + \pi^*(x_3)(e))\pi^{*-1}(x_4)(e).$$

Therefore (12) is true for $k = 2$. Suppose that it true for k . Then with the same reasoning, we have that

$$F_{k+1} = F_{k+1}(x_1, \dots, x_{k-1}, x_k, x_{k+1}, x_{k+2}, x_{k+3}) = F_k(x_1, \dots, x_{k-1}, X_k, X_{k+1}, X_{k+2}),$$

with

$$X_k = \pi(x_k) \left((e + x_{k+1})^{-1} \right),$$

$$X_{k+1} = -\pi^* \left((e + x_{k+1})^{-1} \right) \pi(x_{k+1}) \left(x_{k+2} (e + x_{k+2})^{-1} \right),$$

and

$$X_{k+2} = \pi^* \left((e + \pi^*(x_{k+2})(e))^{-1} \right) (x_{k+3}).$$

Inserting this in $F_k(x_1, \dots, x_{k-1}, X_k, X_{k+1}, X_{k+2})$ as defined in (12), we have in particular that $[x_{i+1}, \dots, X_k]$ is nothing but $[x_{i+1}, \dots, x_k, x_{k+1}]$. We also have

$$\begin{aligned} \pi^{*-1}(X_k) \pi^{*-1}(X_{k+1}) &= -\pi^{*-1}(x_k) \pi^{*-1} \left((e + x_{k+1})^{-1} \right) \pi^{-1} \left((e + x_{k+1})^{-1} \right) \\ &\quad \pi^{*-1}(x_{k+1}) \pi^{*-1}(x_{k+2}) \pi^{*-1} \left((e + \pi^*(x_{k+2})(e))^{-1} \right) \\ &= -\pi^{*-1}(x_k) P(e + x_{k+1}) \pi^{*-1}(x_{k+1}) \pi^{*-1}(x_{k+2}) \\ &\quad \pi^{*-1} \left((e + \pi^*(x_{k+2})(e))^{-1} \right), \end{aligned}$$

and that

$$\pi^{*-1}(X_{k+2})(e) = \pi^{-1} \left((e + \pi^*(x_{k+2})(e))^{-1} \right) \pi^{*-1}(x_{k+3})(e)$$

Thus

$$\begin{aligned} F_{k+1} &= F_{k+1}(x_1, \dots, x_{k-1}, x_k, x_{k+1}, x_{k+2}, x_{k+3}) \\ &= F_k(x_1, \dots, x_{k-1}, X_k, X_{k+1}, X_{k+2}) \\ &= (-1)^{k+1} \pi(x_1) \left(\prod_{i=2}^k \pi^{*-1}(x_i) P(e + [x_{i+1}, \dots, x_{k+1}]) \right) \\ &\quad \pi^{*-1}(x_{k+1}) \pi^{*-1}(x_{k+2}) \\ &\quad \pi^{*-1} \left((e + \pi^*(x_{k+2})(e))^{-1} \right) P(u_k(x_1, \dots, x_{k-1}, X_k, X_{k+1})) \\ &\quad \pi^{-1} \left((e + \pi^*(x_{k+2})(e))^{-1} \right) \pi^{*-1}(x_{k+3})(e). \end{aligned}$$

Denoting

$$g = \pi^{*-1} \left((e + \pi^*(x_{k+2})(e))^{-1} \right) P(u_k(x_1, \dots, x_{k-1}, X_k, X_{k+1})) \pi^{-1} \left((e + \pi^*(x_{k+2})(e))^{-1} \right),$$

It remains to verify that

$$g = P(u_{k+1}(x_1, \dots, x_{k-1}, x_k, x_{k+1}, x_{k+2})).$$

In fact,

$$\begin{aligned} u_k(x_1, \dots, x_{k-1}, X_k, X_{k+1}) = & \\ e - \pi^* \left((e + \pi^*(x_{k+2})(e))^{-1} \right) \pi^*(x_{k+2}) \pi^*(x_{k+1}) \pi \left((e + x_{k+1})^{-1} \right) & \\ \left(e - \pi^* \left((e + x_{k+1})^{-1} \right) \pi^*(x_k) P \left((e + [x_k, x_{k+1}])^{-1} \right) (e + [x_k, x_{k+1}]) + \right. & \\ \sum_{i=1}^{k-3} (-1)^{i+1} \pi^* \left((e + x_{k+1})^{-1} \right) \pi^*(x_k) P \left((e + [x_k, x_{k+1}])^{-1} \right) & \\ \left. \left(\prod_{j=1}^i \pi^*(x_{k-j}) P \left((e + [x_{k-j}, \dots, x_{k+1}])^{-1} \right) \right) (e + [x_{k-i}, \dots, x_{k+1}]) \right) & \end{aligned}$$

Using the fact that

$$\pi^{*-1}(y)P(\pi^*(y)(x))\pi^{-1}(y) = P(x), \quad \text{and} \quad \pi^*(x^{-1})(x) = e$$

we obtain that

$$\begin{aligned} g &= P \left(e + \pi^*(x_{k+2})(e) - \pi^*(x_{k+2}) \pi^*(x_{k+1}) \pi \left((e + x_{k+1})^{-1} \right) \right. \\ &\quad \left(e - \pi^* \left((e + x_{k+1})^{-1} \right) \pi^*(x_k) P \left((e + [x_k, x_{k+1}])^{-1} \right) (e + [x_k, x_{k+1}]) + \right. \\ &\quad \sum_{i=1}^{k-3} (-1)^{i+1} \pi^* \left((e + x_{k+1})^{-1} \right) \pi^*(x_k) P \left((e + [x_k, x_{k+1}])^{-1} \right) \\ &\quad \left. \left(\prod_{j=1}^i \pi^*(x_{k-j}) P \left((e + [x_{k-j}, \dots, x_{k+1}])^{-1} \right) \right) (e + [x_{k-i}, \dots, x_{k+1}]) \right) \Big) \\ &= P \left(e + \pi^*(x_{k+2})(e) - \pi^*(x_{k+2}) \pi^*(x_{k+1}) P \left((e + x_{k+1})^{-1} \right) ((e + x_{k+1})) \right. \\ &\quad \left. + \pi^*(x_{k+2}) \pi^*(x_{k+1}) P \left((e + x_{k+1})^{-1} \right) \pi^*(x_k) P \left((e + [x_k, x_{k+1}])^{-1} \right) (e + [x_k, x_{k+1}]) + \right. \\ &\quad \sum_{i=1}^{k-3} (-1)^{i+2} \pi^*(x_{k+2}) \pi^*(x_{k+1}) P \left((e + x_{k+1})^{-1} \right) \pi^*(x_k) P \left((e + [x_k, x_{k+1}])^{-1} \right) \\ &\quad \left. \left(\prod_{j=1}^i \pi^*(x_{k-j}) P \left((e + [x_{k-j}, \dots, x_{k+1}])^{-1} \right) \right) (e + [x_{k-i}, \dots, x_{k+1}]) \right) \Big). \end{aligned}$$

Setting $i' = i + 1$ and $j' = j + 1$, with some standard calculation, this is noting but $P(u_{k+1}(x_1, \dots, x_{k-1}, x_k, x_{k+1}, x_{k+2}))$. ■

Theorem 4.4 *The sequence (w_k) defined in (7) is decreasing.*

Proof. Form the definition, we have that

$$[e, x_1, x_2, \dots, x_k] = \pi(e) \left((e + [x_1, x_2, \dots, x_k])^{-1} \right),$$

so that

$$[e, x_1, x_2, \dots, x_k]^{-1} = e + [x_1, x_2, \dots, x_k].$$

Hence

$$\begin{aligned} w_k^{-1} &= (-1)^{k+1} ([x_1, x_2, \dots, x_k] - [x_1, x_2, \dots, x_{k+1}])^{-1} \\ &= (-1)^{k+1} \left([e, x_1, x_2, \dots, x_k]^{-1} - [e, x_1, x_2, \dots, x_{k+1}]^{-1} \right)^{-1} \\ w_k^{-1} - w_{k+1}^{-1} &= (-1)^{k+1} \left([e, x_1, x_2, \dots, x_k]^{-1} - [e, x_1, x_2, \dots, x_{k+1}]^{-1} \right)^{-1} \\ &\quad - (-1)^{k+2} \left([e, x_1, x_2, \dots, x_{k+1}]^{-1} - [e, x_1, x_2, \dots, x_{k+2}]^{-1} \right)^{-1} \\ &= (-1)^{k+1} \left\{ \left([e, x_1, x_2, \dots, x_k]^{-1} - [e, x_1, x_2, \dots, x_{k+1}]^{-1} \right)^{-1} \right. \\ &\quad \left. + \left([e, x_1, x_2, \dots, x_{k+1}]^{-1} - [e, x_1, x_2, \dots, x_{k+2}]^{-1} \right)^{-1} \right\} \\ &= (-1)^{k+2} F_{k+1}(e, x_1, \dots, x_{k+2}) \\ &= (-1)^{k+1} (-1)^{k+2} \left(\prod_{i=1}^{k-1} \pi^{*-1}(x_i) P(e + [x_{i+1}, \dots, x_k]) \right) \\ &\quad \pi^{*-1}(x_k) \pi^{*-1}(x_{k+1}) P(u_{k+1}(e, x_1, \dots, x_{k+1})) \pi^{*-1}(x_{k+2})(e) \end{aligned}$$

Therefore

$$w_k^{-1} - w_{k+1}^{-1} = - \left(\prod_{i=1}^{k-1} \pi^{*-1}(x_i) P(e + [x_{i+1}, \dots, x_k]) \right) \pi^{*-1}(x_k) \pi^{*-1}(x_{k+1}) P(u_{k+1}(e, x_1, \dots, x_{k+1})) \pi^{*-1}(x_{k+2})(e). \quad (13)$$

As for all v in the cone Ω , the automorphisms $\pi(v)$, and $P(v)$ are in the group G , they conserve Ω , we deduce that

$$w_k^{-1} - w_{k+1}^{-1} \in -\Omega,$$

which is equivalent to

$$w_k - w_{k+1} \in \Omega.$$

Hence the sequence (w_k) is decreasing. ■

Next, we give an other important intermediary result. Using the notation above, for a sequence (x_n) in the cone Ω , we define the following element of G :

$$Q_k = \left(\prod_{i=1}^{k-1} \pi^{*-1}(x_i) P(e + [x_{i+1}, \dots, x_k]) \right) \pi^{*-1}(x_k) \pi^{*-1}(x_{k+1}) P(u_{k+1}(e, x_1, \dots, x_{k+1})) \quad (14)$$

Note that according to (13), we have

$$w_{k+1}^{-1} - w_k^{-1} = Q_k(x_{k+2}^{-1}). \quad (15)$$

Proposition 4.5 *Let y be an element of Ω . Then for all $k \geq 2$,*

$$Q_k^*(y) \geq \pi^{-1}(x_1)(y)$$

Proof. We have that

$$Q_k^* = P(u_{k+1}(e, x_1, \dots, x_{k+1})) \pi^{-1}(x_{k+1}) \pi^{-1}(x_k) \left(\prod_{i=1}^{k-1} P(e + [x_{k+1-i}, \dots, x_k]) \pi^{-1}(x_{k-i}) \right)$$

Using the fact that $P(x) = \pi(x)\pi^*(x)$, we obtain that Q_k^* is equal to the composition of the elements of G :

$$\pi^*(e + [x_2, \dots, x_k]) \pi^{-1}(x_1). \quad (16)$$

then the automorphisms

$$\pi^*(e + [x_{k+1-i}, \dots, x_k]) \pi^{-1}(x_{k-i}) \pi(e + [x_{k-i}, \dots, x_k]), \quad (17)$$

for $i = 1, \dots, k-2$, and

$$\pi^{-1}(x_k) \pi(e + [x_k]) \quad \text{and} \quad P(u_{k+1}(e, x_1, \dots, x_{k+1})) \pi^{-1}(x_{k+1}). \quad (18)$$

Concerning the terms in (18), we have

$$\begin{aligned} \pi^{-1}(x_k) \pi(e + [x_k]) &= \pi(\pi^{-1}(x_k)(e + [x_k])) \\ &= \pi(\pi^{-1}(x_k)(e) + e). \end{aligned}$$

On the other hand, according to its definition (see Theorem 4.3), we can write $u_{k+1}(e, x_1, \dots, x_{k+1}) = e + \pi^*(x_{k+1})(e + v)$ where v is in Ω . Thus

$$\begin{aligned}
P(u_{k+1}(e, x_1, \dots, x_{k+1})) \pi^{-1}(x_{k+1}) &= P(e + \pi^*(x_{k+1})(e + v)) \pi^{-1}(x_{k+1}) \\
&= \pi(e + \pi^*(x_{k+1})(e + v)) \\
&\quad \pi^*(e + \pi^*(x_{k+1})(e + v)) \pi^{-1}(x_{k+1}) \\
&= \pi(e + \pi^*(x_{k+1})(e + v)) \\
&\quad \pi^*(\pi^{*-1}(x_{k+1})(e + \pi^*(x_{k+1})(e + v))) \\
&= \pi(e + \pi^*(x_{k+1})(e + v)) \\
&\quad \pi^*(\pi^{*-1}(x_{k+1})(e + e + v)).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\pi^{-1}(x_k) \pi(e + [x_k]) &= \pi(\pi^{-1}(x_k)(e + [x_k])) \\
&= \pi(\pi^{-1}(x_k)(e) + e).
\end{aligned}$$

For the terms in (17),

$$\begin{aligned}
&\pi^*(e + [x_{k+1-i}, \dots, x_k]) \pi^{-1}(x_{k-i}) \pi(e + [x_{k-i}, \dots, x_k]) \\
&= \pi^*(e + [x_{k+1-i}, \dots, x_k]) \pi^{-1}(x_{k-i}) \pi(e + \pi(x_{k-i})(e + [x_{k+1-i}, \dots, x_k])^{-1}) \\
&= \pi^*(e + [x_{k+1-i}, \dots, x_k]) \pi(\pi^{-1}(x_{k-i})(e) + (e + [x_{k+1-i}, \dots, x_k])^{-1}) \\
&= \pi(\pi^*(e + [x_{k+1-i}, \dots, x_k]) \pi^{-1}(x_{k-i})(e) + e),
\end{aligned}$$

where in the last equality, we have used the fact that $\pi^*(x)(x^{-1}) = e$.

Having done this, we see that to get $Q_k^*(y)$, we first apply $\pi^{-1}(x_1)$ to y , then we apply automorphisms which are either of the form $\pi(e + z)$ or of the form $\pi^*(e + z)$. Using the fact that for all z and y in Ω , we have that

$$\pi(e + z)(y) - y \in \Omega, \quad \pi^*(e + z)(y) - y \in \Omega \quad \text{and} \quad P(e + z)(y) - y \in \Omega,$$

in other word,

$$\pi(e + z)(y) > y, \quad \pi^*(e + z)(y) > y \quad \text{and} \quad P(e + z)(y) > y,$$

we deduce that given y in Ω , we have that for all $k \geq 2$,

$$Q_k^*(y) > \pi^{-1}(x_1)(y).$$

■

Corollary 4.6 *For all y in Ω , there exists a positive random variable C such that for all k ,*

$$\|Q_k^*(y)\| > C$$

Proof. As $Q_k^*(y) > \pi^{-1}(x_1)(y)$, then $(Q_k^*(y))^2 > (\pi^{-1}(x_1)(y))^2$. In fact

$$(Q_k^*(y))^2 - (\pi^{-1}(x_1)(y))^2 = (Q_k^*(y) - \pi^{-1}(x_1)(y))(Q_k^*(y) + \pi^{-1}(x_1)(y)) \in \Omega.$$

Therefore $\text{tr}((Q_k^*(y))^2) > \text{tr}((\pi^{-1}(x_1)(y))^2)$, which means that $\|Q_k^*(y)\| > \|\pi^{-1}(x_1)(y)\|$. Thus it suffices to take $C = \|\pi^{-1}(x_1)(y)\|$. ■

Besides the general facts that we have established above, for the proof of the main result of the paper concerning the convergence of a random non ordinary continued fraction, we need the following probability result which also appears in [1]. We give it with a slightly different proof.

Proposition 4.7 *Let $(\mathfrak{A}_k)_{k \geq 1}$ be an increasing sequence of σ -fields, and let A_k be in \mathfrak{A}_k , for $k \geq 1$. Suppose that there exist $K > 1$, $0 < k_0 < K$, and a random variable α which is \mathfrak{A}_{k_0} -measurable and valued in $(0, 1)$ such that for all $k \geq K$, $\mathbb{E}(1_{A_{k+1}}|\mathfrak{A}_k) \leq \alpha$. Then*

$$P\left(\bigcap_{k \geq K} A_k\right) = 0$$

Proof. Let us show by induction on $p \geq 1$ that

$$\mathbb{E}(1_{A_K} \dots 1_{A_{K+p}} | \mathfrak{A}_{k_0}) \leq \alpha^p \text{ a.s.}$$

For $p = 1$,

$$\begin{aligned} \mathbb{E}(1_{A_K} 1_{A_{K+1}} | \mathfrak{A}_{k_0}) &= \mathbb{E}((1_{A_K} 1_{A_{K+1}} | \mathfrak{A}_K) | \mathfrak{A}_{k_0}) \\ &= \mathbb{E}(1_{A_K} (1_{A_{K+1}} | \mathfrak{A}_K) | \mathfrak{A}_{k_0}) \\ &\leq \alpha \mathbb{E}(1_{A_K} | \mathfrak{A}_{k_0}) \text{ a.s.} \\ &\leq \alpha \text{ a.s..} \end{aligned}$$

Now suppose that $\mathbb{E}(1_{A_K} \dots 1_{A_{K+p}} | \mathfrak{A}_{k_0}) \leq \alpha^p \text{ a.s..}$ Then

$$\begin{aligned} \mathbb{E}(1_{A_K} \dots 1_{A_{K+p+1}} | \mathfrak{A}_{k_0}) &= \mathbb{E}((1_{A_K} \dots 1_{A_{K+p+1}} | \mathfrak{A}_{K+p}) | \mathfrak{A}_{k_0}) \\ &= \mathbb{E}(1_{A_K} \dots 1_{A_{K+p}} (1_{A_{K+p+1}} | \mathfrak{A}_{K+p}) | \mathfrak{A}_{k_0}) \\ &\leq \alpha \mathbb{E}(1_{A_K} \dots 1_{A_{K+p}} | \mathfrak{A}_{k_0}) \text{ a.s.} \\ &\leq \alpha \alpha^p = \alpha^{p+1} \text{ a.s..} \end{aligned}$$

Having shown that

$$\mathbb{E}(1_{A_K} \dots 1_{A_{K+p}} | \mathfrak{A}_{k_0}) \leq \alpha^p \text{ a.s.},$$

we just need to take the expectation and let $p \rightarrow \infty$ to get the result. ■

We are now in position to state and prove the main result of the paper.

Theorem 4.8 *Let $(x_k)_{k \geq 1}$ be a sequence of independent random variables in the cone Ω with distributions absolutely continuous with respect to the Lebesgue measure and such that for all k , the complementary in Ω of the support of x_k^{-1} is bounded. Suppose that there exists $p \geq 1$, such that for all $i \geq 1$,*

$$\mathfrak{L}(x_i) = \mathfrak{L}(x_{i+p}).$$

Then the continued fraction $[x_1, \dots, x_k]$ is almost surely convergent.

Proof. With the results what we have established above, we need only to show that under these assumptions, the sequence (w_k) defined in (7) converges almost surely to 0, or equivalently that for all y in Ω , the sequence $(\langle w_k^{-1}, y \rangle)_{k \geq 1}$ diverges almost surely. in order to do so, we will show that

$$P\left(\left\{\langle w_{k+1}^{-1} - w_k^{-1}, y \rangle \xrightarrow[k \rightarrow \infty]{} 0\right\}\right) = 0$$

Let \mathfrak{A}_k be the σ -field generated by the random variables x_1, \dots, x_{k+1} , and define for $\varepsilon > 0$,

$$A_k = \{\langle w_k^{-1} - w_{k-1}^{-1}, y \rangle \leq \varepsilon\}.$$

Then $A_k \in \mathfrak{A}_k$, and we have

$$\left\{\langle w_{k+1}^{-1} - w_k^{-1}, y \rangle \xrightarrow[k \rightarrow \infty]{} 0\right\} \subset \bigcup_{n>0} \bigcap_{k \geq n} A_k.$$

Given that $w_{k+1}^{-1} - w_k^{-1} = Q_k(x_{k+2}^{-1})$, with Q_k defined in (14) we have

$$\begin{aligned} A_{k+1} &= \{\langle Q_k(x_{k+2}^{-1}), y \rangle \leq \varepsilon\} \\ &= \{\langle x_{k+2}^{-1}, Q_k^*(y) \rangle \leq \varepsilon\} \\ &= \left\{\langle x_{k+2}^{-1}, \frac{Q_k^*(y)}{\|Q_k^*(y)\|} \rangle \leq \frac{\varepsilon}{\|Q_k^*(y)\|}\right\}. \end{aligned}$$

Let S be the unit sphere in V , and define for a random variable z ,

$$h_z(a) = \sup_{\theta \in S} P(|\langle z, \theta \rangle| \leq a),$$

then we have that

$$\begin{aligned}
\mathbb{E}(1_{A_{k+1}}|\mathfrak{A}_k) &= P\left(\left\{\langle x_{k+2}^{-1}, \frac{Q_k^*(y)}{\|Q_k^*(y)\|} \rangle \leq \frac{\varepsilon}{\|Q_k^*(y)\|}\right\}|\mathfrak{A}_k\right) \\
&\leq h_{x_{k+2}^{-1}}\left(\frac{\varepsilon}{\|Q_k^*(y)\|}\right), \text{ because } Q_k^*(y) \text{ is } \mathfrak{A}_2 - \text{measurable} \\
&\leq h_{x_{k+2}^{-1}}\left(\frac{\varepsilon}{C}\right).
\end{aligned}$$

Denoting $\alpha = \max\left(h_{x_1^{-1}}\left(\frac{\varepsilon}{C}\right), \dots, h_{x_p^{-1}}\left(\frac{\varepsilon}{C}\right)\right)$, then according to the assumptions on the distributions of the x_i and on the support of x_i^{-1} , we have that $\alpha < 1$, and that

$$\mathbb{E}(1_{A_{k+1}}|\mathfrak{A}_k) \leq \alpha.$$

Therefore using Proposition 4.7 with $k_0 = 2$, we deduce that

$$P\left(\bigcap_{k \geq n} A_k\right),$$

which is the desired result. ■

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